

On a representation of Humbert's double hypergeometric series Φ_3 in a series of Gauss's ${}_2F_1$ function

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Abstract

Very recently a new series representation of Humbert's double hypergeometric series Φ_3 in series of Gauss's ${}_2F_1$ function was given by one of us. The aim of this short research note is to provide an alternative proof of the result. A few interesting special cases are also given.

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1. Introduction

The double hypergeometric series, defined by Humbert [1, 2] are the following :

$$\Psi_2(a; b, c; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}}{(b)_n (c)_k} \frac{x^n y^k}{n! k!}, \quad (1.1)$$

$$\Phi_3(b; c; x, y) = \sum_{n,k=0}^{\infty} \frac{(b)_n}{(c)_{n+k}} \frac{x^n y^k}{n! k!} \quad (1.2)$$

The double series (1.1) and (1.2) converge absolutely for all $x, y \in \mathbb{C}$. Very recently, Manako [3] established a few results for the series Φ_2 , Φ_3 and Ψ_2 out of which, two results are given here :

For $b, c \neq 0, -1, -2, \dots$

$$\Psi_2(b; b, c; x, y) = \exp(x + y) \Phi_2(c - b; c; -y, xy) \quad (1.3)$$

and for $b, c \neq 0, -1, -2, \dots$ and $|x| \neq 0$

$$\Psi_2(a; b, c; x, y) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} {}_2F_1 \left[\begin{matrix} -k, -k - b + 1 \\ c \end{matrix} ; \frac{y}{x} \right] \frac{x^k}{k!}. \quad (1.4)$$

In the same paper, using (1.3) and (1.4), Manako [3] established the following new result for Φ_3 in terms of series of ${}_2F_1$,

$$\Phi_3(b; c; x, y) = \exp \left(x + \frac{y}{x} \right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{y}{x} \right)^k {}_2F_1 \left[\begin{matrix} -k, -k - b + 1 \\ c \end{matrix} ; \frac{x^2}{y} \right]. \quad (1.5)$$

In 2013, Rathie[4] obtained the following result for the series Φ_2 :

For $c \neq 0, -1, -2, \dots$

$$\Phi_2(a, b; c; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} {}_2F_1 \left[\begin{matrix} -m, b \\ 1 - a - m \end{matrix} ; \frac{y}{x} \right] \frac{x^m}{m!}. \quad (1.6)$$

and discussed some special cases.

The aim of this short research note is to establish (1.5) by another method. In the end, we consider some interesting special cases.

2 Derivation of (1.5)

In order to derive (1.5), let us denote its right-hand side by S and expressing ${}_2F_1$ as a series, we have

$$S = \exp \left(x + \frac{y}{x} \right) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-k)_m (-k - c + b + 1)_m (-1)^k}{(c)_m m! k!} x^{2m-k} y^{k-m}$$

Using

$$\frac{(-k)_m}{k!} = \frac{(-1)^m}{(k-m)!}$$

we have

$$S = \exp \left(x + \frac{y}{x} \right) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-k - c + b + 1)_m}{(c)_m m!} \frac{(-1)^{k+m} x^{2m-k} y^{k-m}}{(k-m)!}$$

Replacing k by $k + m$, and using the result[5, Lemma 10, equ. 2, p-57]

$$S = \exp\left(x + \frac{y}{x}\right) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-k-m-c+b+1)_m}{(c)_m m!} \frac{(-1)^k x^{m-k} y^k}{k!}$$

Using $(-k-m-c+b+1)_m = (-1)^m (k+c-b)_m$, we have

$$S = \exp\left(x + \frac{y}{x}\right) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (k+c-b)_m}{(c)_m m!} \frac{(-1)^k x^{m-k} y^k}{k!}$$

which can be written as

$$S = \exp\left(x + \frac{y}{x}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{y}{x}\right)^k \sum_{m=0}^{\infty} \frac{(-1)^m (k+c-b)_m}{(c)_m m!} x^m$$

Now, summing up the inner series, we have

$$\begin{aligned} S &= \exp\left(x + \frac{y}{x}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{y}{x}\right)^k {}_1F_1\left[\begin{matrix} k+c-b \\ c \end{matrix}; -x\right] \\ &= \exp\left(\frac{y}{x}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{y}{x}\right)^k}{k!} \left\{ \exp(x) {}_1F_1\left[\begin{matrix} k+c-b \\ c \end{matrix}; -x\right] \right\}. \end{aligned}$$

Using Kummer's first transformation [5]

$$\exp(x) {}_1F_1\left[\begin{matrix} c-b \\ c \end{matrix}; -x\right] = {}_1F_1\left[\begin{matrix} b \\ c \end{matrix}; x\right]. \quad (2.1)$$

we have

$$S = \exp\left(\frac{y}{x}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{y}{x}\right)^k}{k!} {}_1F_1\left[\begin{matrix} b-k \\ c \end{matrix}; x\right].$$

Again, expressing ${}_1F_1$ as a series, we have

$$S = \exp\left(\frac{y}{x}\right) \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(-\frac{y}{x}\right)^k}{k!} \frac{(b-k)_n}{(c)_n} \frac{x^n}{n!}$$

Since $(b-k)_k = \frac{(b)_n (1-b)_n}{(1-b-n)_k}$, we have, therefore

$$S = \exp\left(\frac{y}{x}\right) \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(1-b)_k}{(1-b-n)_k} \frac{\left(-\frac{y}{x}\right)^k}{k!}$$

Summing up the inner series

$$\begin{aligned} S &= \exp\left(\frac{y}{x}\right) \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{x^n}{n!} {}_1F_1\left[\begin{matrix} 1-b \\ 1-b-n \end{matrix}; -\frac{y}{x}\right] \\ &= \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{x^n}{n!} \left\{ \exp\left(\frac{y}{x}\right) {}_1F_1\left[\begin{matrix} 1-b \\ 1-b-n \end{matrix}; -\frac{y}{x}\right] \right\}. \end{aligned}$$

Using (2.1), we have

$$S = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{x^n}{n!} {}_1F_1 \left[\begin{matrix} -n \\ 1-b-n \end{matrix} ; \frac{y}{x} \right].$$

Expressing ${}_1F_1$ as a series

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(b)_n}{(c)_n} \frac{(-n)_m}{(1-b-n)_m} \frac{x^n \left(\frac{y}{x}\right)^m}{m! n!}$$

Using $(-n)_m = \frac{(-1)^m n!}{(n-m)!}$

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(b)_n (-1)^m}{(c)_n (1-b-n)_m} \frac{x^n \left(\frac{y}{x}\right)^m}{m! (n-m)!}$$

Changing n to $n+m$ and using [5, Lemma 10, equ. 2, p-57], we have

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b)_{n+m} (-1)^m y^m x^n}{(c)_{n+m} (1-b-n-m)_m m! n!}$$

Using $(1-b-n-m)_m = \frac{(-1)^m (b)_{n+m}}{(b)_n}$

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b)_n x^n y^m}{(c)_{n+m} m! n!}$$

Finally, using definition (1.2), we have

$$S = \Phi_3(b; c; x, y)$$

This completes the proof of (1.5).

3 SPECIAL CASES

In this section, we shall mention two interesting special cases of our results (1.5).

In (1.5), if we take $y = x^2$, we have

$$\Phi_3(b; c; x, x^2) = \exp(2x) \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \left(-\frac{y}{x}\right)^k {}_2F_1 \left[\begin{matrix} -k, -k-c+b+1 \\ c \end{matrix} ; 1 \right]. \quad (3.1)$$

The ${}_2F_1$ appearing on the right-hand side of (3.1) can be evaluated with the help of classical Gauss's summation theorem [6]

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; 1 \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (3.2)$$

provided $\text{Re}(c - a - b) > 0$.

and we get, after some simplification, the following new reduction formula

$$\Phi_3(b; c; x, x^2) = \exp(2x) {}_2F_2 \left[\begin{matrix} c - \frac{b}{2}, c - \frac{b}{2} - \frac{1}{2} \\ c, 2c - b - 1 \end{matrix} ; -4x \right] \quad (3.3)$$

Further, in (3.3), if we take $c = 2b$, we have

$$\Phi_3(b; 2b; x, x^2) = \exp(2x) {}_2F_2 \left[\begin{matrix} \frac{3b}{2}, \frac{3b-1}{2} \\ 2b, 3b-1 \end{matrix} ; -4x \right]$$

and using (1.4) after simplification, we get

$$\Psi_2(b; b, 2b; x, x) = {}_2F_2 \left[\begin{matrix} \frac{3b}{2}, \frac{3b-1}{2} \\ 2b, 3b-1 \end{matrix} ; 4x \right] \quad (3.4)$$

which is a special case of the following result

$$\Psi_2(b; b, c; x, x) = {}_3F_3 \left[\begin{matrix} a, \frac{c+b}{2}, \frac{c+b-1}{2} \\ b, c, c+b-1 \end{matrix} ; 4x \right]$$

given by Burchnell and Chaundy [6, 7], also recorded in [1, 2].

Similarly, other results can also be obtained.

Remark :

In 2015, Choi and Rathie [8] obtained explicit expressions of

$$\Phi_2(a, a+i; c; x, -x)$$

and

$$\Psi_2(a, c; c+i; x, -x)$$

each for $i = 0, \pm 1, \pm 2, \dots, \pm 5$.

and deduced interesting summation formulas.

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